Pat O'Sullivan

Mh4718 Week 8

Week 8

0.1 Solving Differential Equations

We will consider only differential equations which can be written in the form:

$$\frac{dy}{dx} = F(x, y).$$

Examples:

- $\dfrac{dy}{dx} = y,$ that is $F(x, y) = y$.
- $\dfrac{dy}{dx} = \sqrt{1 - y^2},$ that is $F(x, y) = \sqrt{1 - y^2}$.
- $\dfrac{dy}{dx} = \dfrac{2y}{x},$ that is $F(x, y) = \dfrac{2y}{x}$.

If $f(x)$ is a solution for the equation $\dfrac{dy}{dx} = F(x, y)$ then

$$\frac{df(x)}{dx} = F(x, f(x)).$$

In general there is more than one solution to a well formed differential equation. For example, $y = e^x$ and $y = 3e^x$ are both solutions of the d.e. $\dfrac{dy}{dx} = y$. In fact, it is easy to see there are infinitely many solutions to this d.e. In order to pick out a unique solution we need to specify what are known as *initial values* for a solution. That is, we must specify one value which a solution must have at a particular point.

For example if we said that we want a solution of $\dfrac{dy}{dx} = y$ with $y(0) = 1$ then we see that $y = e^x$ satisfies the equations and has $y(0) = 1$ but $y = 3e^x$ does

not have the required so-called initial values.
Note that specifying initial values is the same as specifying a point which must be on the graph of the solution.
(0,1) is on the graph of $y = e^x$ but is not on the graph of $3e^x$.

A differential equation $\dfrac{dy}{dx} = F(x, y)$ together with initial values $y(x_0) = y_0$ is called an *initial value problem.* (IVP)

If certain conditions are fulfilled, an inital value problem has precisely one solution. We have the following theorem:

Theorem 0.1

If F and $\dfrac{\partial}{\partial y}F(x, y)$ are continuous in a rectangle containing the point (x_0, y_0) then the initial value problem

$$\frac{dy}{dx} = F(x, y), \quad y(x_0) = y_0$$

has a unique solution $y = f(x)$ defined in some open interval around x_0.

The initial value problem supplies us with enough information to determine the Taylor expansion around the intitial value x_0 for the unique solution.

Example 0.2

(i) We already know that the IVP $\dfrac{dy}{dx} = y, y(0) = 1$ has solution $y = e^x$. but we can construct the Taylor series around 0 for this solution from the IVP as follows:

$$y(x) = y(0) + y^{(1)}(0)x + y^{(2)}(0)\frac{x^2}{2!} + y^{(3)}(0)\frac{x^3}{3!} + \dots$$

Using this notation the IVP is

$$y^{(1)}(x) = y(x), y(0) = 1.$$

And so we have

$$\begin{aligned} y(0) &= 1 \\ y^{(1)}(x) = y \Rightarrow y^{(1)}(0) &= y(0) = 1 \\ y^{(2)}(x) = y^{(1)}(x) = y \Rightarrow y^{(2)}(0) &= y(0) = 1 \\ y^{(3)}(x) = y^{(2)}(x) = y \Rightarrow y^{(3)}(0) &= y(0) = 1 \end{aligned}$$

Continuing like this we have

$$y(0) = 1, y^{(1)}(0) = 1, y^{(2)}(0) = 1, y^{(3)}(0) = 1, y^{(4)} = 1\ldots$$

and so we get the Taylor series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

which we recognise as the Taylor series for e^x.
If we didn't know it already we could now conclude that the solution $y(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$

(ii) Consider the IVP

$$\frac{dy}{dx} = \frac{2y}{x}, \quad y(1) = 1.$$

We can attempt to solve this using a Taylor series again.

$$y(1) = 1,$$

$$y^{(1)}(x) = \frac{2y(x)}{x} \Rightarrow y^{(1)}(1) = \frac{2y(1)}{1} = \frac{2}{1} = 2$$

$$y^{(2)}(x) = \frac{2y^{(1)}x - 2y}{x^2} = \frac{2\frac{2y}{x}x - 2y}{x^2} = \frac{4y - 2y}{x^2} = \frac{2y}{x^2}$$

$$y^{(2)}(1) = \frac{2}{1} = 2$$

$$y^{(3)}(x) = \frac{2y^{(1)}x^2 - 4yx}{x^4} = \frac{2\frac{2y}{x}x^2 - 4yx}{x^4} = \frac{4yx - 4yx}{x^4} = 0$$

Therefore we get the finite Taylor expansion

$$y(x) = 1 + 2(x-1) + 2\frac{(x-1)^2}{2!} = x^2$$